

Dynamic asset trees and portfolio analysis

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Abstract

The minimum spanning tree, based on the concept of ultrametricity, is constructed from the correlation matrix of stock returns and provides a meaningful economic taxonomy of the stock market. In order to study the dynamics of this asset tree we characterize it by its normalized length and by the mean occupation layer, as measured from an appropriately chosen center. We show how the tree evolves over time, and how it shrinks particularly strongly during a stock market crisis. We then demonstrate that the assets of the optimal Markowitz portfolio lie practically at all times on the outskirts of the tree. We also show that the normalized tree length and the investment diversification potential are very strongly correlated.

Keywords: portfolio optimization, time dependency of stock correlations, minimum spanning tree.

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Portfolio optimization is one of the basic tools of hedging in a risky and extremely complex financial environment. Many attempts have been made to solve this central problem starting from the classical approach of Markowitz [1] to more sophisticated treatments including spin glass type studies [2]. In all of these attempts, correlations between asset prices play a crucial role. A closely related problem is that of economic taxonomy. In his recent paper [3], Mantegna suggested the study the clustering of companies by using the correlation matrix of asset returns such that a simple transformation of the correlations into distances produces a connected graph. In the graph the nodes are the companies and the ‘distances’ between them are obtained from the correlation coefficients and the clusters of companies are identified by means of the minimum spanning tree. It turned out that in this way the hierarchical structure of the financial market could be identified in accordance with the results obtained by an independent clustering method based on Potts super-paramagnetic transitions [4]. In another paper by Bonanno et al. [5], the time evolution of stock indices was studied and significant changes in the world economy were identified by using appropriate time horizons and the minimum spanning tree clustering method. The hierarchical structure explored by the minimum spanning tree also seemed to give information about the influential power of the companies. The network of influence was recently investigated by means of a time-dependent correlation method [6]. Some other attempts have been made to understand the structure of correlation matrices in a highly random setting using the theory of random matrices [7].

In this paper, we study the minimum spanning tree determined from correlations between stock returns and call it an ‘asset tree’. Although this asset tree can reveal a great deal about the taxonomy of the market at a given time, it only represents a snapshot of an evolving complex system. This evolution is a reflection of the changing power structure in the market and manifests the passing of different products and product generations, new technologies, management teams, alliances and partnerships, amongst many other things. This is why exploring the asset tree *dynamics* can provide us new insights to the market. Here, by studying the time evolution of the asset tree we show that although the structure of the tree

changes with time, the companies of the optimal Markowitz portfolio are always on its outer leaves. We also study the robustness of the tree topology and the consequences of the market events on its structure. The minimum spanning tree, as a strongly pruned representative of asset correlations is found to be robust and descriptive of stock market events.

We start our analysis by assuming that there are N assets with price $P_i(t)$ for asset i at time t . Then the logarithmic return of stock i is $r_i(t) = \ln P_i(t) - \ln P_i(t-1)$, which for a certain consecutive sequence of trading days forms the return vector \mathbf{r}_i . In order to characterize the synchronous time evolution of stocks, we use the equal time correlation coefficients between stocks i and j defined as

$$\rho_{ij} = \frac{\langle \mathbf{r}_i \mathbf{r}_j \rangle - \langle \mathbf{r}_i \rangle \langle \mathbf{r}_j \rangle}{\sqrt{[\langle \mathbf{r}_i^2 \rangle - \langle \mathbf{r}_i \rangle^2][\langle \mathbf{r}_j^2 \rangle - \langle \mathbf{r}_j \rangle^2]}}, \quad (1)$$

where $\langle \dots \rangle$ indicates a time average over the trading days included in the return vectors. These correlation coefficients forming an $N \times N$ matrix with $-1 \leq \rho_{ij} \leq 1$, is then transformed to an $N \times N$ distance matrix with elements $d_{ij} = \sqrt{2(1 - \rho_{ij})}$, such that $2 \geq d_{ij} \geq 0$, respectively. The d_{ij} s fulfill the requirements of distances, even those of ultrametricity [3]. We now use the distance matrix to determine the minimum spanning tree (MST) of the distances, denoted by \mathbf{T} , which is a simply connected graph that connects all the N nodes of the graph with $N-1$ edges such that the sum of all edge weights, $\sum_{(i,j) \in \mathbf{T}} d_{ij}$, is minimum. It should be noted that in constructing the minimum spanning tree, we are effectively reducing the information space from $N(N-1)/2$ separate correlation coefficients to $N-1$ tree edges.

The dataset we have used in this study consists of daily closure prices for 116 stocks of the S&P 500 index [8], which were obtained from the Yahoo website [9]. The time period of this data extends from the beginning of 1982 to the end of 2000 including a total of 4787 price quotes per stock, after the removal of a few days due to incomplete data. We divide this data into M windows $t = 1, 2, \dots, M$ of width T corresponding to the number of

daily returns included in the window. Different windows overlap with each other, the extent of which is dictated by the window step length parameter δT , describing the displacement between two consecutive windows, measured also by the number of trading days. The choice of the window width is a trade-off between too noisy and too smoothed data for small and large window widths, respectively. In our studies, T was set to be typically between 500 and 1500 trading days, i.e., 2 and 6 years, and δT to one month including about 21 trading days. This is in accordance with the suggestions of the Basel committee [10].

In order to study the temporal state of the market we define the *normalized tree length* as

$$L(t) = \frac{1}{N-1} \sum_{d_{ij} \in \mathbf{T}^t} d_{ij}, \quad (2)$$

where t denotes the time at which the tree is constructed, and $N-1$ is the number of edges present in the MST. To characterize the position of companies in the graph, i.e., the layers on which the different nodes are located at a given time, we introduce the concept of a *central node*. Although there is arbitrariness in the choice of the central node, we propose that it is central in the sense that any change in its price strongly affects the course of events in the market on the whole. Thus the central node would be the company which is most strongly connected to its nearest neighbors in the tree. With this choice the sum of the correlation coefficients calculated for the incident edges would be maximum, and/or have the highest *vertex degree* (the number of edges which are incident with the vertex). It is also noted that one can have either a static (fixed at all times) or a dynamic (continuously updated) central node, without considerable effects on the results. In our studies, General Electric (GE) was chosen as the central node, since for about 70% of the period considered it was the most connected node. A typical asset tree is shown in Figure 1, where it is evident that companies become clustered by business sectors.

Figures 2 (a) and (b) show how the normalized tree length L and the mean correlation coefficient, defined as $\bar{\rho} = \frac{1}{N(N-1)/2} \sum \rho_{ij}$, where we consider only the non-diagonal and independent ρ_{ij} , evolve with time. The two curves, indeed, look like mirror images, which is corroborated by the fact that the correlation coefficient is -0.96 , indicating that the minimum spanning tree is a strongly reduced representative of the whole correlation matrix and bears the essential information about asset correlations. As further evidence that the MST retains the salient features of the stock market, it is noted that the 1987 market crash can be quite accurately seen in Figure 2. The two sides of the ridge actually converge as a result of extrapolating the window width $T \rightarrow 0$ [11]. In Figure 2 (a), the mean correlation of stocks is very high during the crash. This is because the market forces act strongly on all the stocks and force the market to behave in a unified way. Figure 2 (b) also strengthens this fact: $L(t)$ decreases indicating that the nodes on the graph are drawn closer together. In order to characterize the spread of nodes on the graph, we introduce the quantity of *mean occupation layer* as

$$l(t) = \frac{1}{N} \sum_{i=1}^N \text{lev}(v_i^t), \quad (3)$$

where $\text{lev}(v_i)$ denotes the level of vertex v_i in relation to the central node, whose level is taken to be zero. We find that $l(t)$ reaches a very low value at the time of a market crisis (see Figure 3).

Next, we apply the above discussed concepts and measures to portfolio analysis. We consider a minimum risk Markowitz portfolio $P(t)$ with the asset weights w_1, w_2, \dots, w_N . In the Markowitz portfolio optimization scheme financial assets are characterized by their average return and risk, both determined from historical price data, where risk is measured by the standard deviation of returns. The aim is to optimize the asset weights so that the overall portfolio risk is minimized for a given portfolio return [12]. In the minimum spanning

tree framework, the task is to determine how the assets are located with respect to the central node. Intuitively, we expect the weights to be distributed on the outskirts of the graph. In order to describe what happens, we define a single measure, the *weighted portfolio layer* as

$$l_P(t) = \sum_{i \in P} w_i \text{lev}(v_i^t), \quad (4)$$

where we have the constraint $w_i \geq 0$ for all i , since we assume that there is no short-selling.

Figure 3 shows the behaviour of the mean layer $l(t)$ and the weighted minimum risk portfolio layer $l_P(t)$. We find that the portfolio layer is higher than the mean layer practically at all times. The difference in layers depends to a certain extent on the window width: for $T = 500$ it is about 0.76 and for $T = 1000$ about 0.97. As the stocks of the minimum risk portfolio are found on the outskirts of the graph, we expect larger graphs (higher L) to have greater *diversification potential*, i.e., the scope of the stock market to eliminate specific risk of the minimum risk portfolio. In order to look at this, we calculated the mean-variance frontiers for the ensemble of 116 stocks using $T = 500$ as the window width. In Figure 2 (c), we plot the level of portfolio risk as a function of time, and find a striking similarity between the risk curve and the curves of the mean correlation coefficient $\bar{\rho}$ and normalized tree length L of Figures 2 (a) and (b). The correlation between the risk and $\bar{\rho}$ is 0.82, while the correlation between the risk and L is -0.90 . Therefore, the latter result explains the diversification potential of the market better.

Finally, in order to investigate the robustness of the minimum spanning tree topology, we define the survival ratio of tree edges (fraction of edges is found common in both graphs) at time t as

$$\sigma_t = \frac{1}{N-1} |E^t \cap E^{t-1}|.$$

In this E^t refers to the set of edges of the graph at time t , \cap is the intersection operator and $|\dots|$ gives the number of elements in the set. Under normal circumstances, the graphs at two consecutive time windows t and $t + 1$ (for small values of δT) should look very similar. Whereas some of the differences can reflect real changes in the asset taxonomy, others may simply be due to noise. We find that as $\delta T \rightarrow 0$, $\sigma_t \rightarrow 1$ [11], indicating that the graphs *are* stable in the limit, and hence our portfolio analysis is justified.

In summary, we have studied the dynamics of asset trees and applied it to portfolio analysis. We have shown that the tree evolves over time and have found that the normalized tree length decreases and remains low during a crash, thus implying the shrinking of the asset tree particularly strongly during a stock market crisis. We have also found that the mean occupation layer fluctuates as a function of time, and experiences a downfall at the time of market crisis due to topological changes in the asset tree. As for the portfolio analysis, it was found that the stocks included in the minimum risk portfolio tend to lie on the outskirts of the asset tree: on average the weighted portfolio layer is about 1 level higher, or further away from the central node, than mean occupation layer for window width of four trading years. The correlation between the risk and the mean correlation was found to be quite strong, though not as strong as the correlation between the risk and the normalized tree length. Thus it can be concluded that the diversification potential of the market is very closely related to the behaviour of the normalized tree length.

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Figure Captions

Fig. 1 : A typical asset taxonomy (minimum spanning tree) graph connecting the examined 116 stocks of the S&P 500 index. The graph was produced using four-year window width and it is centered on January 1, 1998. Business sectors are indicated according to Forbes, <http://www.forbes.com>. In this graph, General Electric (GE) was used as a central node and eight layers can be identified.

Fig. 2 : Plots of (a) the mean correlation coefficient $\bar{\rho}$, (b) the normalized tree length L and (c) the risk of the minimum risk portfolio, as functions of time. The risk is determined with weight limits of zero lower bound (no short-selling) and unit upper bound (any asset may constitute the entire portfolio). For all plots the window width is $T = 500$, i.e., two trading years.

Fig. 3 : Plots of mean occupation layer l and weighted portfolio layer l_P as functions of time. This plot is based on the window width $T = 1000$, i.e., four trading years.

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- Basic Materials
- Capital Goods
- ◆ Conglomerates
- ▲ Consumer/Cyclical
- ▼ Consumer/Non-Cyclical
- * Energy
- Financial
- Healthcare
- ◆ Services
- ▲ Technology
- ▼ Transportation
- * Utilities





